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# The Implicit Spin Magnetic and Electric Moments of an Electron Moving in Accordance with the Lorentz-Dirac Equation

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*Abstract*

Spin and magnetic-electric moment effects are shown to be implicit in the Lorentz-Dirac equation of motion of a point charge. A new condition for a non-radiative motion emerges.

The various classical relativistic theories of a spinning point electron have been reviewed and compared by Nyborg (1962a, b, c, 1964). All authors associate with the electron an empirical spin angular momentum tensor, $\sigma^{\alpha\beta}$, and all authors except Corben (1961a, b, 1962) introduce also an empirical magnetic-electric moment tensor, $\mu^{\alpha\beta}$, which is related to $\sigma^{\alpha\beta}$ by $\mu^{\alpha\beta} = (ge/2mc)\sigma^{\alpha\beta}$. While the addition of empirical spin parameters in an ad hoc fashion may be legitimate for a non-relativistic theory, it cannot be correct for a relativistic theory. In a relativistic theory spin effects are already implicit in the equations, and the addition of empirical spin parameters becomes entirely superfluous. This will be true both for quantum mechanics and classical electrodynamics. In the non-relativistic Schrödinger quantum mechanics Pauli spin matrices must be introduced as an 'extra'; in the relativistic Dirac quantum theory of the electron all spin effects emerge automatically from the equations. Even in the old quantum theory of Bohr and Sommerfeld it was found that the energies of the stationary states were in accord with Dirac's wave mechanics and with observation although no spin parameters were introduced; the spin-orbit energy was implicit.

Only Corben has shown any recognition of the general requirement that spin effects should be implicit in relativistic classical equations of motion. Corben chooses $g = 0$ on the grounds that agreement of the cross section for scattering of light by a free electron with the Klein-Nishina formula and with the observed cross-section is possible only with this choice. The magnetic and electric moments of the electron arise in Corben's theory from Zitterbewegung, which is due to a term, $\ddot{\sigma}^{\alpha\beta} v_\beta/c^2$, in the equation of motion. Here we shall show that when radiative reaction is included there is no need to introduce $\sigma^{\alpha\beta}$ empirically; it is already implicit in the Lorentz-Dirac equation of motion. It will turn out that the assumption that the electric moment of the electron vanishes in the rest frame, adopted as an auxiliary condition by Corben, is untenable for discussion of Zitterbewegung although it will apply to the ordinary motion.

We shall adopt the Minkowski metric in the form,

$$c^2\,d\tau^2 = (dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2 \tag{1}$$

where $x^0 = ct$. Proper time, $\tau$, and ordinary time, $t$, are related by $d\tau = dt/\gamma$, where $\gamma = [1-(v^2/c^2)]^{-1/2}$. A dot will denote derivative with respect to $\tau$, so that particle velocity and acceleration are $v^\alpha = \dot{x}^\alpha$ and $a^\alpha = \dot{v}^\alpha$ respectively.

The Lorentz-Dirac equation of motion is

$$ma^\alpha - \Gamma^\alpha = F^\alpha = eF^{\alpha\beta}\,v_\beta/c \tag{2}$$

where $F^\alpha$ is the Lorentz force exerted on an electron of charge $e$ and mass $m$ by the retarded electromagnetic field $F^{\alpha\beta}$, and where $\Gamma^\alpha$ is the 4-force of radiative reaction. The components of the antisymmetric tensor $F^{\alpha\beta}$ are

$$\{F^{01}, F^{02}, F^{03}\} \equiv -\mathbf{E}; \qquad \{F^{23}, F^{31}, F^{12}\} \equiv -\mathbf{B} \tag{3}$$

$\Gamma^\alpha$ is given by

$$\Gamma^\alpha = k(v^\alpha\,\dot{a}^\beta - \dot{a}^\alpha\,v^\beta)\,v_\beta/c^2 = -k\dot{a}^\alpha + Iv^\alpha/c^2 \tag{4}$$

where $k = 2e^2/3c^3$ and where

$$I = -ka^\beta a_\beta = k\gamma^6[\mathbf{a}^2 - (\mathbf{v}\times\mathbf{a}/c)^2] \qquad (\mathbf{a} = d\mathbf{v}/dt) \tag{5}$$

We define an antisymmetric tensor, $\dot{\sigma}^{\alpha\beta}$, by

$$\dot{\sigma}^{\alpha\beta} = k(v^\alpha\,a^\beta - a^\alpha\,v^\beta) \tag{6}$$

whose components are

$$\{\dot{\sigma}^{01}, \dot{\sigma}^{02}, \dot{\sigma}^{03}\} \equiv -\dot{\boldsymbol{\xi}} = kc\gamma^3\,\mathbf{a}; \qquad \{\dot{\sigma}^{23}, \dot{\sigma}^{31}, \dot{\sigma}^{12}\} \equiv -\dot{\boldsymbol{\sigma}} = k\gamma^3\,\mathbf{v}\times\mathbf{a} \tag{7}$$

We define also an effective 4-momentum by

$$P^\alpha = mv^\alpha - \dot{\sigma}^{\alpha\beta}\,v_\beta/c^2 = mv^\alpha + ka^\alpha \tag{8}$$

and an effective mass, $M$, by

$$M^2\,c^2 = P^\alpha P_\alpha = m^2\,c^2 + \tfrac{1}{2}\dot{\sigma}^{\alpha\beta}\,\dot{\sigma}_{\alpha\beta}/c^2 = m^2\,c^2 - k\,I \tag{9}$$

Note that $mc^2 = P^\alpha\,v_\alpha$.

From equations (6) and (8) it follows straightforwardly that

$$\dot{\sigma}^{\alpha\beta} = (\dot{\sigma}^{\alpha\lambda}\,v^\beta - v^\alpha\,\dot{\sigma}^{\beta\lambda})\,v_\lambda/c^2 \tag{10}$$

$$= v^\alpha P^\beta - P^\alpha\,v^\beta \tag{11}$$

Using equation (1) and assuming that $\dot{m} = 0$, one then obtains

$$-\dot{\sigma}^{\alpha\beta} a_\beta = I v^\alpha = (\dot{P}^\lambda v_\lambda) v^\alpha \tag{12}$$

$$F^\alpha = m a^\alpha - \ddot{\sigma}^{\alpha\beta} v_\beta / c^2 = \dot{P}^\alpha - I v^\alpha / c^2 \tag{13}$$

From equations (11) and (13) it follows that

$$\frac{d}{d\tau}(\sigma^{\alpha\beta} + P^\alpha x^\beta - x^\alpha P^\beta) = F^\alpha x^\beta - x^\alpha F^\beta + (v^\alpha x^\beta - x^\alpha v^\beta) I/c^2 \tag{14}$$

We shall find below that $I \to 0$ as a result of Zitterbewegung. Thus this equation expresses that the moment of the external force equals the rate of change of the total angular momentum (spin plus orbital). Hence $\sigma^{\alpha\beta}$ can be identified with the spin angular momentum of the electron.

An auxiliary condition is necessary in order to completely determine the motion of the particle. Often this has been taken to be $\sigma^{\alpha\beta} v_\beta = 0$, which expresses that the electric moment of the electron vanishes in the rest frame. However, if one differentiates this condition and substitutes into equation (10) it follows that $\dot{\sigma}^{\alpha\beta} a_\beta = 0$ and hence $I = 0$. While $I$ may tend to zero for the ultrarelativistic Zitterbewegung it cannot equal zero [equation (9)]. Thus we shall adopt, instead, the auxiliary condition

$$\sigma^{\alpha\beta} = -t_0 \dot{\sigma}^{\alpha\beta} = -t_0 k (v^\alpha a^\beta - a^\alpha v^\beta) \tag{15}$$

where $t_0$ is a constant. Since $\sigma^{\alpha\beta} v_\beta = t_0 c^2 k a^\alpha$, this essentially is the auxiliary condition used by Halbwachs (1959) and by Bohm *et al.* (1960). Notice that both $\sigma^{\alpha\beta}$ and $\dot{\sigma}^{\alpha\beta}$ are proportional to the Thomas precession. From equation (15) it follows that

$$\tfrac{1}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta} = \sigma^2 - \xi^2 = -c^2 t_0^2 k I \tag{16}$$

$$\frac{d}{d\tau}(\sigma^{\alpha\beta}\sigma_{\alpha\beta}) = 2\dot{\sigma}^{\alpha\beta}\sigma_{\alpha\beta} = -2(\sigma^{\alpha\beta}\sigma_{\alpha\beta})/t_0 \tag{17}$$

so that $I$ must satisfy

$$I = I_0 \exp(-2\tau/t_0) \tag{18}$$

We shall find below that $t_0 = k/m = 0{\cdot}62 \times 10^{-23}$ s. Hence in a very short time $I$ decreases to a negligible value. If $I \to 0$, then from equation (9) it follows that $(\mathbf{v} \times \mathbf{a}/c)^2 \to \mathbf{a}^2$, so that the electron must have an ultra-relativistic Zitterbewegung in agreement with quantum theory. Although $\tfrac{1}{2}\sigma^{\alpha\beta}\sigma_{\alpha\beta}$ and hence $-v_\mu \sigma^{\mu\alpha}\sigma_{\alpha\nu} v^\nu$ tends to zero, the magnitude of the spin angular momentum does not tend to zero; $\sigma^{\alpha\beta} v_\beta$ merely tends to a null vector. For the order of magnitude of the spin angular momentum we have

$$|\boldsymbol{\sigma}| = t_0 k \gamma^3 |\mathbf{v} \times \mathbf{a}| \sim t_0 (2e^2/3c^3) \gamma^3 (c^2/t_0) \sim \gamma^3 e^2/c \tag{19}$$

Thus $|\boldsymbol{\sigma}| \sim \hbar$ if $\gamma^3 \sim \hbar c/e^2 = 137$.

With the result $I \to 0$, equation (8) yields $M \to m$, and equations (13) and (14) simplify to familiar forms.

From equations (15) and (13) it follows that

$$\sigma^{\alpha\beta} F_{\alpha\beta} = -2t_0(mI + \tfrac{1}{2}k\dot{I}) = 0 \tag{20}$$

provided that we substitute $t_0 = k/m$ into equation (18). Just as in Dirac's quantum theory there is no explicit interaction between the electromagnetic field and the magnetic and electric moments of the electron. Bearing in mind that $I \to 0$, a constant of the motion is

$$m^2 c^2 \simeq M^2 c^2 = P^\alpha P_\alpha - e\sigma^{\alpha\beta} F_{\alpha\beta}/c \tag{21}$$

If $P^\alpha = (W/c, \mathbf{P})$ this yields

$$W^2 = m^2 c^4 + c^2 P^2 + 2ec(\boldsymbol{\xi}.\mathbf{B} - \boldsymbol{\sigma}.\mathbf{E}) \tag{22}$$

which becomes, in the non-relativistic approximation,

$$W \simeq mc^2 + P^2/2m + (e/mc)(\boldsymbol{\xi}.\mathbf{B} - \boldsymbol{\sigma}.\mathbf{E}) \tag{23}$$

The electron therefore behaves as if it has a magnetic moment, $-e\boldsymbol{\sigma}/mc$, and an electric moment, $e\boldsymbol{\xi}/mc$.

The rate at which a moving charge radiates energy commonly is assumed to be $I$. But $\Gamma^\alpha$ does not necessarily vanish when $I$ vanishes, and conversely. This has led Rohrlich (1960) to suggest $Iv^\alpha$ for radiative reaction. However, $\Gamma^\alpha$ has a more fundamental claim, being the Lorentz force exerted on a charge due to the difference of its retarded and advanced fields (Dirac, 1938). Since $\Gamma^\alpha v_\alpha = 0$, the components of $\Gamma^\alpha$ can be expressed as $(\gamma\boldsymbol{\Gamma}.\mathbf{v}/c, \gamma\boldsymbol{\Gamma})$. The time-like component is the rate of working of $\boldsymbol{\Gamma}$ with respect to proper time divided by $c$. The rate of radiation of energy by a charge, $R$, should therefore be given by

$$R = -\mathbf{v}.\boldsymbol{\Gamma} = -c\Gamma^0/\gamma \tag{24}$$

In terms of the three dimensional vectors, $\mathbf{v} = d\mathbf{r}/dt$, $a = d\mathbf{v}/dt$, and $\mathbf{b} = d\mathbf{a}/dt$, the components of $\Gamma^\alpha$ are

$$\gamma\boldsymbol{\Gamma}.\mathbf{v}/c = -k\gamma^5 \mathbf{v}.\mathbf{h}/c; \qquad \gamma\boldsymbol{\Gamma} = -k\gamma^3[\mathbf{h} + \gamma^2(\mathbf{v}.\mathbf{h}/c^2)\mathbf{v}] \tag{25}$$

where

$$\mathbf{h} = \mathbf{b} + 3\gamma^2(\mathbf{v}.\mathbf{a}/c^2)\mathbf{a} = \gamma^{-3}\frac{d}{d\tau}(\gamma^3\mathbf{a}) \tag{26}$$

Thus

$$R = k\gamma^4 \mathbf{v}.\mathbf{h} \tag{27}$$

A motion will be non-radiative if $\mathbf{h}$ either vanishes or is normal to $\mathbf{v}$. As shown by Rohrlich (1965) in another connection, the vanishing of $\mathbf{h}$ is the condition for constant acceleration relative, not to the laboratory frame, but to the instantaneous rest frame of the charge. The vanishing of $\mathbf{h}$ implies $\gamma^3\mathbf{a}$ = constant. If $\mathbf{v}$ is parallel to $\mathbf{a}$ initially then it will remain so, and this result can be further integrated to yield motion under a constant force, $d(\gamma\mathbf{v})/dt$ = constant. Motion under a constant force parallel to the velocity now is non-radiative.

Using expression (2) for $\Gamma^\alpha$ with $I \to 0$, equation (24) yields the approximate expression,

$$R \simeq k \frac{d}{dt}(\gamma^4 \mathbf{v} . \mathbf{a}) \tag{28}$$

The energy radiated between points $A$ and $B$ on the world line of the charge is then given by

$$\int_B^A R\,dt = [k\gamma^4 \mathbf{v} . \mathbf{a}]_B^A \tag{29}$$

For perfectly periodic motion the radiated energy vanishes. Thus electrons in stationary atomic orbits should not now radiate. Whereas $I = 0$ demanded that $\mathbf{a} = 0$ (apart from Zitterbewegung), the condition $R = 0$ does not demand $\mathbf{a} = 0$.

In essence the classical description of electron spin depends on the Thomas precession. The connection between the quantum mechanical spin matrix operators and the Thomas precession has been elucidated elegantly by Furry (1955). While the consequences of the auxiliary condition (15) are interesting, it is stressed that other auxiliary conditions are not ruled out.

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